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Probabilistic Approach to Aircraft Performance Enhancement in Atmospheric Turbulence

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The aircraft response to atmospheric turbulence and the related ride quality problem are revisited through a probabilistic approach. The aircraft performance is characterized by means of the probability of exceeding an acceleration value belonging to a fixed range of interest. This formulation constitutes an important improvement with respect to a classical approach where only mean information is employed, such as the root mean square acceleration value in one or more fuselage stations. Key features of the proposed algorithm are 1) the possibility of modeling some parameters (like the aircraft derivatives) as random variables with given probability density functions, 2) the possibility of finding the best (in a probabilistic sense) ride quality controller in a candidate set, 3) the availability of information about the most important parameter that must be controlled to improve the design, and 4) the possibility of obtaining more rational information about the controller robustness with respect to a standard approach. A detailed example demonstrates the effectiveness of the proposed methodology.

Introduction

FROM a ride quality point of view, the aircraft response to atmospheric turbulence generates an unwanted random motion that decreases the comfort of pilot and passengers. It is well known that the problem is essentially due to the low human tolerance for even moderate acceleration levels. The ride quality level is usually quantified by means of a scalar variable that is strictly connected to the rms acceleration experienced in one or more fuselage stations. This parameter is attractive because it is possible to obtain the rms value with standard numerical techniques in an efficient way.[1] Another, slightly different approach has also been proposed,[2] but it is still based on "mean" response characteristics. Despite the simplicity of the given problem solution, a number of important questions cannot properly be investigated. First, a mean value does not take into account the effect of an instantaneous acceleration level that, in effect, is a more appropriate variable to be considered for evaluating the level of human discomfort. Second, there is no systematic methodology to quantify the effect of parameter uncertainties on the ride quality level. Both of these aspects may be investigated by means of a probabilistic problem formulation, which is the aim of the present paper. We propose to complete the mean value information with the knowledge of the probability of exceeding a given range of vertical acceleration levels, of particular significance from a ride quality viewpoint. We show how it is possible to obtain such information in the case in which a fixed number of parameters are not deterministic, but modeled as independent random variable with given distribution functions. In particular, we investigate how the uncertainties on the aerodynamic derivatives influence the ride quality characteristics. One interesting result is that the performance of different controllers may be easily compared. To this end, a family of controllers achieving the same rms value as the vertical acceleration for the nominal system is obtained through an eigenstructure assignment approach combined with a classical optimization procedure. Then, among the controllers of this family, the most appropriate one is chosen as that which minimizes the probability of encountering the given range of vertical acceleration levels. There are also other important implications of our approach. First, the controller robustness is given in a probabilistic sense with the advantage of a better physical interpretation and less conservative information as compared to the usual robustness verification, where only sufficient conditions can be checked. Second, the sensitivity of various parameters can be quantified, thus giving fundamental information to the engineer as to where additional information is needed to improve the design.

System Model

The aircraft dynamics are described by the following linear, time-invariant system:

$$\dot{\boldsymbol{x}} = A_0\boldsymbol{x} + B_{0w}\boldsymbol{w}_g + B_{0u}\boldsymbol{u} \tag{1}$$

$$\boldsymbol{z} = C_{0z}\boldsymbol{x} + D_{zw}\boldsymbol{w}_g + D_{zu}\boldsymbol{u} \tag{2}$$

$$\boldsymbol{y} = C_{0y}\boldsymbol{x} + D_{yw}\boldsymbol{w}_g + D_{yu}\boldsymbol{u} \tag{3}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the vector containing the states of the system, $\boldsymbol{u} \in \mathbb{R}^m$ denotes the control variable, $\boldsymbol{w}_g \in \mathbb{R}^r$ contains the gust velocity components, $\boldsymbol{y} \in \mathbb{R}^{p_y}$ is the vector of measured signals, and $\boldsymbol{z} \in \mathbb{R}^{p_z}$ contains the regulated output signals. The atmospheric turbulence is described by means of a linear system driven by white noise:

$$\dot{\boldsymbol{d}}_g = A_g\boldsymbol{d}_g + B_g\boldsymbol{\eta} \tag{4}$$

$$\boldsymbol{w}_g = C_g\boldsymbol{d}_g \tag{5}$$

and the matrices in Eqs. (4) and (5) are chosen in accordance with a Dryden model. The vector $\boldsymbol{d}_g \in \mathbb{R}^s$ contains the states of the atmospheric disturbances and $\boldsymbol{\eta} \in \mathbb{R}^q$ is white noise with constant intensity. Obviously, Eqs. (1–5) may be easily combined by defining an augmented state vector

$$\boldsymbol{\xi} := \begin{Bmatrix} \boldsymbol{x} \\ \boldsymbol{d}_g \end{Bmatrix} \tag{6}$$

with the following result

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} A_0 & B_{0w}C_g \\ 0 & A_g \end{bmatrix}\boldsymbol{\xi} + \begin{bmatrix} 0 \\ B_g \end{bmatrix}\boldsymbol{\eta} + \begin{bmatrix} B_{0u} \\ 0 \end{bmatrix}\boldsymbol{u} \tag{7}$$

$$\boldsymbol{z} = [C_{0z} \quad D_{zw}C_g]\boldsymbol{\xi} + D_{zu}\boldsymbol{u} \tag{8}$$

$$\boldsymbol{y} = [C_{0y} \quad D_{yw}C_g]\boldsymbol{\xi} + D_{yu}\boldsymbol{u} \tag{9}$$

Our aim is to measure the ride quality level of the open-loop system and to compare it to that obtained by employing a suitable control law, in the form

$$\boldsymbol{u} = K\boldsymbol{y} \tag{10}$$

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where K is a constant matrix. In this latter case the aircraft dynamics assume the form

$$\dot{\xi} = A\xi + G\eta \tag{11}$$

$$z = C\xi \tag{12}$$

where

$$A := \begin{bmatrix} A_0 + B_{0u}KC_{0y} & B_{0w}C_g \\ 0 & A_g \end{bmatrix}, \qquad G := \begin{bmatrix} 0 \\ B_g \end{bmatrix}$$

$$C := [C_{0z} + D_{zu}KC_{0y} \quad D_{zw}C_g] \tag{13}$$

Note that the rms acceleration value at a fixed fuselage station of the aircraft is equal to $E[zz^T]$, where $E[\cdot]$ is the expectation operator. The evaluation of $E[zz^T]$ is a simple matter. Indeed, the following result holds[1,2]:

$$E[zz^T] = CXC^T \tag{14}$$

and X is the symmetric matrix solution of the following Lyapunov equation:

$$AX + XA^T + GG^T = 0 \tag{15}$$

We incidentally observe that, by defining $P(s) := C(sI - A)^{-1}G$ (s being the Laplace operator), $E[zz^T]$ turns out also to be equal to $\|P\|_2^2$, where $\|\cdot\|_2$ is the H_2 norm of the system.

Probabilistic Environment

In this section we describe the probabilistic approach we used to address the ride quality problem. An important aspect of a probabilistic formulation lies in the necessity of properly choosing the aircraft configuration in a set of different controllers that equally satisfy to the same (classical) requirement of the rms value of vertical acceleration at a fixed fuselage station. Among the controllers that achieve the same rms acceleration value, the *best* one is that which reduces the probability of encountering a given range of vertical acceleration levels. To check the controllers against this requirement, a probabilistic framework is necessary. This will also allow us to take into account not only the randomness of the turbulence phenomenon, but also the uncertainties on the aerodynamic derivatives and how they affect the aircraft performance.

We start with a general formulation of the problem, and, to this end, we first define the probability that the event of interest takes place as

$$P = \int_D f_X(\boldsymbol{x}, \boldsymbol{p})\,\mathrm{d}\boldsymbol{x} \tag{16}$$

where $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)^T$ is a deterministic parameter vector and $f_X(\boldsymbol{x})$ is the probability density function of the random variable vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$. The domain of integration of $f_X(\boldsymbol{x})$ is $D = \{g(\boldsymbol{x}, \boldsymbol{p}) \le 0\}$, where $g(\boldsymbol{x}, \boldsymbol{p})$ is the limit state function having the properties that $g(\boldsymbol{x}, \boldsymbol{p}) > 0$ denotes the favorable states, $g(\boldsymbol{x}, \boldsymbol{p}) = 0$ is the limit state and $g(\boldsymbol{x}, \boldsymbol{p}) \le 0$ denotes the unfavorable states. The domain region D is referred to as the admissible region. From a practical standpoint, Eq. (16) can be rather difficult to evaluate, especially when the uncertainty space is of high dimensions or when the limit state function is of a complex nature. For these reasons, analytical results are hard to find. However, efficient methods have been developed in the last two decades in the field of structural reliability engineering to approximate the evaluation of P. These are referred to as first- or second-order reliability methods, and they have been introduced with the aim of evaluating the probability of failure of a given structure. Nevertheless, the methodology is fully general and may be applied to any problem formulated in the form of Eq. (16).

In essence, these methods reduce the cumbersome task of integration to that of locating the most probable point in the admissible region. The idea is that of using a set of simple algebraic manipulations to transform the problem in such a way that exact or asymptotic results may easily be obtained. Also, Monte Carlo schemes may be added to replace or update the approximate results and to quantify the possible error.

The key result of the analytical approach is this: If the variables X_i are jointly normal distributed and the limit state function is an hyperplane in the form

$$g(\boldsymbol{X}) = a_0 + \sum_{i=1}^{n} a_i X_i := a_0 + \boldsymbol{a}^T\boldsymbol{X} \tag{17}$$

then an analytical solution of Eq. (16) is given by[3]

$$P = \Phi(-\beta) \tag{18}$$

where

$$\beta := \frac{a_0 + \boldsymbol{a}^T E[\boldsymbol{X}]}{\sqrt{\boldsymbol{a}^T C_X \boldsymbol{a}}} \tag{19}$$

$E[\boldsymbol{X}]$ is the vector of the expected values, C_X is the covariance matrix of $\boldsymbol{X}$, and $\Phi(\cdot)$ is the standard normal probability distribution function. Note that if the basic variables X_i are independent standard normally distributed (in this case we will denote them by $\boldsymbol{U}$), β is still given by Eq. (19) with $E[\boldsymbol{U}] = 0$ and $C_U = I$, where I is the identity matrix.

Suppose now that the random variables X_i are jointly normal distributed but the limit state function is not linear in $\boldsymbol{X}$. In this case the problem may be approximated by means of the following fundamental result due to Hasofer and Lind[4]:

$$P \approx \Phi(-\beta) \tag{20}$$

with

$$\beta = \|\boldsymbol{u}^*\| := \min\{\|\boldsymbol{u}\|\} \qquad \text{for} \qquad \{\boldsymbol{u} \mid g(\boldsymbol{u}) \le 0\} \tag{21}$$

where $\|\cdot\|$ denotes the Euclidean norm and $\boldsymbol{u}^*$ is referred to as the β point (see Fig. 1).

However, in general, the random variables X_i are not normally distributed. In that case it has been shown[5,6] that this difficulty can be eliminated by introducing a suitable one-to-one transformation, for example, T, of the basic random variables $\boldsymbol{X}$ into uncorrelated and standard normally distributed random variables $\boldsymbol{U}$, that is,

$$T : \boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T \longrightarrow \boldsymbol{U} = (U_1, U_2, \ldots, U_n)^T \tag{22}$$

such that, on this transformed basis, the following result is asymptotically exact[7]:

$$P \sim \Phi(-\beta) \cdot \prod_{i=1}^{n-1} (1 - \kappa_i\beta)^{-\frac{1}{2}} = \Phi(-\beta) \cdot C \tag{23}$$

The symbol $\sim$ in Eq. (23) means that the calculation of P is exact in the limit when $\beta \longrightarrow \infty$ (and, correspondingly, $P \longrightarrow 0$). Also, C is a second-order correction factor. We stress that the transformation T is always possible for continuous random variables with invertible distribution functions.

For practical purposes Eq. (23) may be used whenever β is greater than one (which corresponds to $P < 0.841$), but, under some additional mild restrictions, it is possible to extend the result also to the case where $|\beta| \le 1$ (Ref. 8).

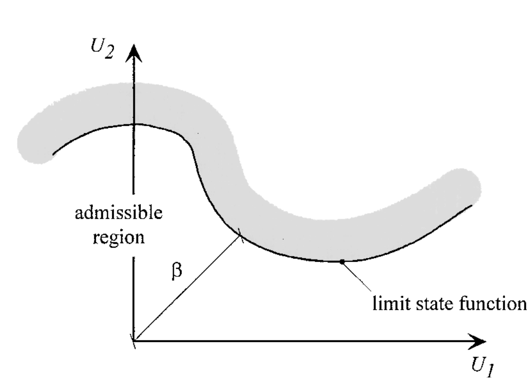

Fig. 1 Example of limit state function and corresponding definition of β.

The preceding theory has been extended by Hohenbichler et al.[9] to include the case where the limit state function is given as the intersection of several individual (componential) state functions. Accordingly, one has

$$P\left(\bigcap_{j=1}^{k}\{g_j(\boldsymbol{u})\} \leq 0\right) \sim \Phi_k(-\boldsymbol{b}; \boldsymbol{R}) \cdot C$$

where

$$\beta = \|\boldsymbol{u}^*\| := \min\{\|\boldsymbol{u}\|\} \qquad \text{for} \qquad \left\{\boldsymbol{u} \mid \bigcap_{j=1}^{k}\{g(\boldsymbol{u}) \leq 0\}\right\} \tag{24}$$

is now the joint β point, $\Phi_k(.;.)$ is the multivariate standard normal integral whose value may be approximated by means of the schemes outlined by Gollwitzer and Rackwitz,[10] C is a second-order correction factor similar to the one in Eq. (23), $\boldsymbol{b}$ is the vector of distances of the limit surface linearized in the joint β point, and $\boldsymbol{R} = \{\boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j\}$ is the matrix of the correlation coefficients of the linearized margins $Z_i := \boldsymbol{\alpha}_i^T \boldsymbol{u} + \boldsymbol{\alpha}_i^T \boldsymbol{u}^* = \boldsymbol{\alpha}_i^T \boldsymbol{u} + b_i$.

Similar results can be derived for unions of limit domains and for (minimal) unions of intersections to which any combination of sets can be reduced. Of great importance is that Schall et al.[11] succeeded in deriving schemes that can also include equality constraints so that conditional probabilities can be determined as

$$P(D \mid B) = P(D \cap B)/P(B)$$

where D is any limit function domain and B any condition given in the form of a combination of arbitrary inequalities or equalities.

Probabilistic Ride Quality

The main advantage of using the preceding methodology to solve the general problem formulated in the form of Eq. (16) lies in the capability of executing a large amount of parameter studies in an high dimensional space, which would not be possible otherwise. In the present paper the preceding approach has been specialized to the event of encountering a vertical gust leading to an unacceptable (from a passenger comfort viewpoint) vertical acceleration level. The result is that the probability of exceeding some given acceleration levels may be evaluated, and with such a value at hand, the ride quality effectiveness of different controllers may be compared.

The problem may be tackled as follows: Given the rms acceleration level required for the controlled system, find a family of controllers, for example, $\mathcal{S}$, that nominally attain this value and such that the flight quality requirements are met. The uncertainties of the aerodynamic derivatives are then taken into account and modeled as random variables with appropriate distribution functions. The optimal controller is that which belongs to $\mathcal{S}$ and minimizes the probability of encountering a given acceleration level belonging to the range of interest. The limit state function [i.e., the function $g(\boldsymbol{x}, \boldsymbol{p})$ that delimits the admissible region of the system] is obtained through a response surfaces approach.[12] In our case we generate random samples of the vector $\boldsymbol{x}$ and, for each realization of the controlled system (i.e., for a fixed value of $\boldsymbol{p}$), the maximum vertical acceleration encountered in a fixed fuselage location is evaluated by simulation; finally, the limit state function is obtained by approximating, in a least-square sense, the generated data with a surface, typically of linear or quadratic type.

The computational procedure can be subdivided into these points:

1) Define the rms value of the vertical acceleration that the nominal aircraft should achieve.

2) Find a family of controllers that nominally attain this value and meet the aircraft flight qualities. Let $\mathcal{S}$ represent such a family.

3) Model the aerodynamic derivatives that present uncertainties as random variables with appropriate probability distribution functions.

4) For all of the controllers belonging to $\mathcal{S}$, find a relation that expresses the maximum vertical acceleration $a_{\max}$ encountered as a function of the aerodynamic derivatives modeled as random variables.

5) Define the range $a_{\max}$ of interest.

6) Define the limit state function for the problem at hand by combining the information of the earlier two points.

7) Transform the problem from the physical state to a normalized space (i.e., transform the random variables into independent normal distributed random variables) and, accordingly, transform the limit state function.

8) Compute the probability of exceeding the vertical acceleration level of interest.

9) Identify the most appropriate controller as that which minimizes the probability of exceeding the vertical acceleration level of interest.

Case Study

The data used in the current example are taken from Ref. 2 where the short-period dynamics of a DC-8-type aircraft with symmetric aileron deflection is described. The aircraft is trimmed in a cruise condition at an altitude of 33,000 ft with a Mach number equal to 0.84. In our case study a standard deviation of 10 ft/s^2 is considered for the Dryden spectrum. In open-loop conditions, the rms acceleration value at the most rearward fuselage station (the reference point) is equal to 7.54 ft/s^2. Note that this is the point where typically the acceleration level is maximum.[2,13] Now, suppose that the requirement is to reduce the acceleration level to less than 5 ft/s^2 and that a set of controllers does exist, regardless of how they may be effectively found. In our example we purchased a specialized strategy: A family of controllers has been generated having the characteristics that all of the controlled systems attain the same rms acceleration value at the reference point. The reason is that with our choice all controllers have been "equalized" and a more meaningful comparison may be made to select the best controller in the set. Because we deal with a quite simple example, the problem of finding a given number of different controllers with the given characteristics is not prohibitive. Indeed, because the aircraft dynamics are described by means of a short-period approximation, the plant is a system with two states and two inputs, and the eigenstructure is completely assignable with a full state static control law. We stress that the simplicity of the control strategy used does not affect the applicability of the proposed probabilistic approach to more complex control philosophies. A standard optimization procedure has been used to force the controlled system to have a prescribed closed-loop rms acceleration value. Accordingly, we have generated a family of four different controllers [in the form of Eq. (10)] such that the rms acceleration value at the reference point is equal to 3 ft/s^2. The four matrices of the controllers are displayed in Table 1, and the corresponding eigenvalues have been summarized in Table 2. Note that all of the cases are acceptable from a flight quality viewpoint.

Because the aerodynamic coefficients are affected by errors, we have chosen to model some aerodynamic derivatives as random variables. In particular Z_w, M_w, $M_{\dot{w}}$, and M_q have been assumed to have a trapezoidal distribution with a uniform variation of ±10% around the mean value and with a 10% linear decrease to zero (see Fig. 2), where x_n is the nominal value and x is the generic aerodynamic

Table 2 Eigenvalues of the controlled aircraft for the four cases under study

Case	Eigenvalue	Damping	Frequency, rad/s
1	$-2.37 \pm j3.06$	0.614	3.87
2	$-3.06 \pm j3.34$	0.675	4.53
3	$-2.93 \pm j3.24$	0.670	4.37
4	$-2.94 \pm j5.37$	0.480	6.13

Table 1 Controller matrix gain for the four cases

Case 1		Case 2		Case 3		Case 4	
−0.0003	0.7013	−0.0016	−1.0000	−0.0019	−0.6779	−0.0020	−1.0000
−0.0039	−4.7719	−0.0011	0.5847	0.0003	−0.4784	−0.0148	1.2226

Table 3 Correspondence between the reliability index β and the probability of exceeding a given vertical acceleration level P

Parameter	Value											
β	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5
P	0.5	0.309	0.159	0.067	0.023	$6.21e-3$	$1.35e-3$	$2.326e-4$	$3.167e-5$	$3.398e-6$	$2.867e-7$	$1.899e-8$

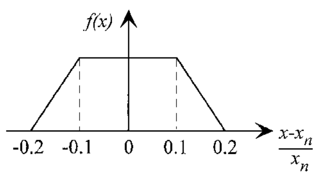

Fig. 2 Probability density function $f(x)$ of the generic aerodynamic derivative modeled as a random variable; x_n is the nominal value.

derivative. As outlined in the preceding section, the limit state function has been obtained through a response surfaces approach, that is, generating random samples of Z_w, M_w, $M_{\dot{w}}$, and M_q from the given trapezoidal distribution. Then, for each realization of the controlled system, (that is, for a fixed value of K) the maximum vertical acceleration encountered at the reference point of the fuselage has been obtained. Finally a function has been fitted to the generated data. Note that the limit state condition has the following structure:

$$g(\boldsymbol{x}, \boldsymbol{p}) - a_{\max} \leq 0 \tag{25}$$

and, in the present example, it assumes the form

$$g(Z_w, M_w, M_{\dot{w}}, M_q, \boldsymbol{p}) - a_{\max} \leq 0 \tag{26}$$

The task of performing the transformation from the physical to the normalized space and then the calculating the probability integral according to the procedure described earlier has been addressed with the aid of the program COMREL, which belongs to the structural reliability software package STRUREL.[14] Programs for reliability analysis of structural, operational, and other systems based on first- and second-order reliability concepts were made available as early as 1976 by the Technical University of Munich. The program package STRUREL consists of four basic modules: COMREL, SYSREL, NASREL, and STATREL. COMREL is a set of programs covering time-invariant and time-variant reliability analysis of individual failure modes (components). SYSREL is a program for system reliability. NASREL is a high-performance finite element program integrated with COMREL. STATREL is a statistical reliability oriented program for basic data analysis. The programs COMREL and SYSREL share a common interface to user-defined modules. Important capabilities of all STRUREL programs are the determination of hazard functions and the calculation of reliability sensitivity and importance measures, for example, with respect to parameters in the stochastic model, deterministic design parameters, or components in the system. Those quantities not only help to identify the most significant basic variables, but can also be part of the input into optimization procedures or other codes.

Returning now to our example, results are shown in Fig. 3 where the behavior of the four different controllers are compared.

Note that for a given vertical acceleration level, corresponding to a horizontal line ideally traced in Fig. 3, the probability of exceeding this level (given from the intersection of this ideal line with the four curves) can be rather different, although the four controllers attain the same rms value. Results are shown in terms of β. In Table 3 the values of β are reported with the corresponding probability of exceeding a given acceleration level according to Eq. (18).

The effect of the uncertainties in the aerodynamic derivatives is clearly shown in Fig. 4 where case 4 (corresponding to a total variation of $\pm 20\%$ around the mean level of all aerodynamic derivatives, see Fig. 2) is compared to the case in which a total variation of only $\pm 10\%$ has been assumed. As one would expect, the probability of the adverse event (in this case the exceeding of an acceleration level), increases when the uncertainties involved in the problem are increased.

In Fig. 5 another interesting result is reported. The different probability levels are shown for the same controller (case 4) for the case where Z_w, M_w, $M_{\dot{w}}$, and M_q are all considered random variables

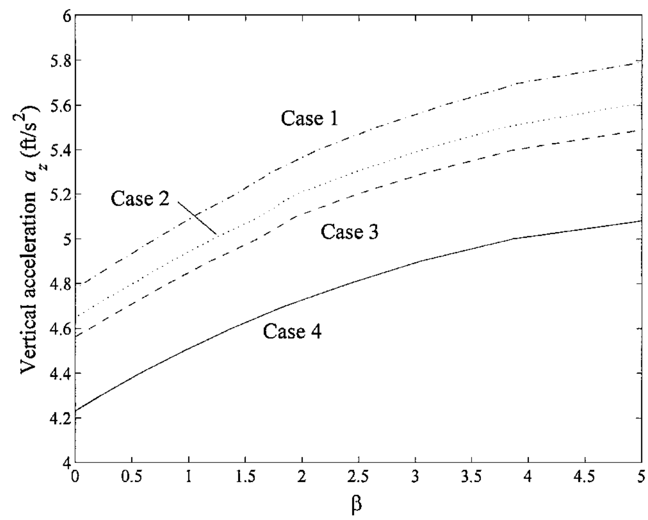

Fig. 3 Performance comparison of the four controllers.

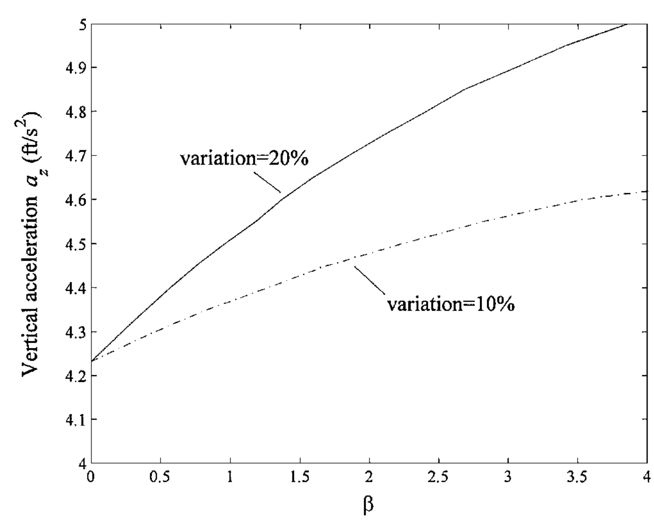

Fig. 4 Effect of different uncertainty values on aircraft performance.

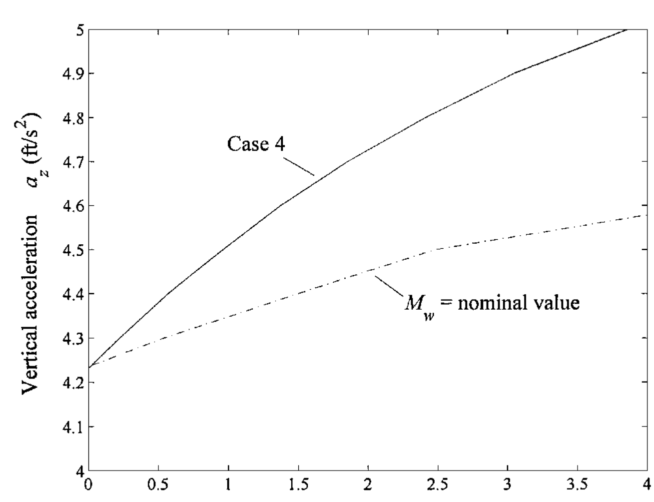

Fig. 5 Effect of M_w on aircraft performance.

compared to the case where M_w is considered constant. The probability levels are quite different between the two cases. As would be expected, paralleling the behavior of Fig. 4, when dealing with an higher number of random variables there is a larger probability of exceeding a given acceleration level. From Fig. 5, the importance of a correct modeling of the random variables involved in the phenomenon is apparent. Also, an estimate can be performed of the conservatism of the assumptions made.

Note that in both Figs. 4 and 5 the curves related to the two different cases cross each other at $\beta = 0$. This result is not casual, but is a precise consequence of the model we used in our formulation. Indeed, the intersection point means that the two limit state functions cross the origin of the normalized space for the same value of the varying parameter, that is, for the same value of a_z. This is not surprising because the origin of the normalized space corresponds to the mean value of the random variables of the problem. Moreover, seeing that the random variables for the case discussed in Fig. 4 have the same mean (recall that only the variation around the mean value is different), it makes sense that the two curves do intersect at the same value of a_z at $\beta = 0$. Finally, because in Fig. 5 M_w is a deterministic quantity, it attains its nominal value; hence, the values assumed by Z_w, M_w, $M_{\dot{w}}$, and M_q are the same at the origin of the normalized space and the same value of a_z is obtained.

A useful concept in the analysis of results is that of elasticity. The influence of a given parameter τ_i on the reliability index β is named elasticity[15] and is defined as

$$e_i := \frac{\partial \beta}{\partial \tau_i} \frac{\tau_i}{\beta}$$

where τ_i is evaluated at the β point. SYSREL provides the elasticity for a deterministic parameter as well as for the mean and standard deviation of a random variable. Analogous results may also be obtained if the distribution function has been defined by means of distribution parameters. Note that elasticities of the standard deviation of random variables are almost always negative because an increase in the standard deviation usually decreases the reliability.

Figure 6 shows the elasticities of the four aerodynamic derivatives that have been modeled as random variables.

When applied to our example, the concept of elasticity may be employed to obtain information about the derivative of the probability of encountering a given vertical acceleration level with respect to the random variables involved. The results are very useful because the designer may easily appreciate which is the random variable (in this case M_w) that has the larger effect on the probability evaluation result. This is obviously the design variable that is most important to try to take under control (i.e., that must be evaluated with lesser uncertainties) to improve the aircraft performance.

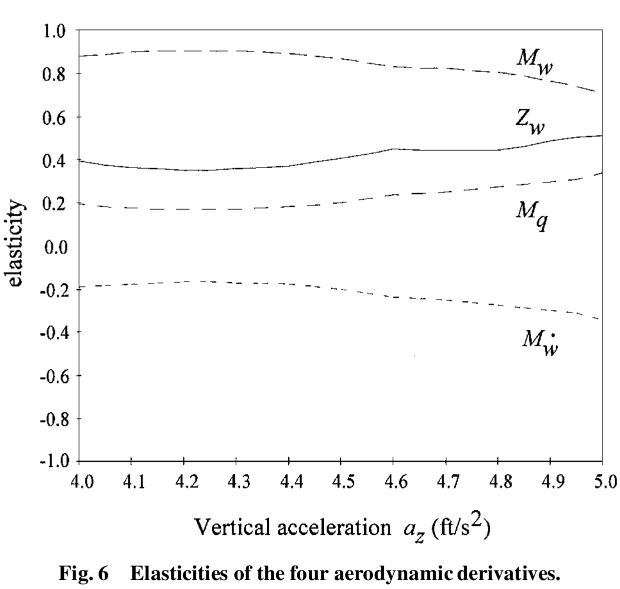

Fig. 6 Elasticities of the four aerodynamic derivatives.

Conclusions

A novel methodology has been proposed for the analysis of the aircraft performance in atmospheric turbulence. The problem is handled in a probabilistic framework by modeling the variables affected by uncertainties as random variables with assigned density functions and by evaluating the minimum distance from the origin of the limit state function. Actually, this result holds true only if the random variables are jointly normal distributed. However, a suitable transformation exists that allows one to approach the problem in the more general case.

Our formulation may be seen as a powerful analysis tool to validate the results that can be obtained with any control law. As a result, fundamental information may be easily obtained during the optimization process of aircraft performance. In particular, given a set of candidate controllers that nominally attain a fixed requirement, the *best* one in the set is obtained as the one that minimizes the probability of exceeding some acceleration levels belonging to a fixed range of interest. We stress that the proposed methodology gives performance information that is less conservative than that usually employed in the control literature because the latter is typically based on sufficient conditions. The sensitivity concept has also been introduced by means of the definition of the elasticity of a variable. Significant information may be obtained from this definition on where (i.e., on what variable) the strengths must be concentrated to improve the design.

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